Section 7.8: Convolution

Section 7.8: Convolution

Definition

Convolution

Definition 9. Let $f(t)$ and $g(t)$ be piecewise continuous on $[0, \infty)$. The **convolution** of $f(t)$ and $g(t)$, denoted $f * g$, is defined by

$$(f * g)(t) := \int_0^t f(t - v) g(v)\, dv\,. \tag{3}$$

Ex 0: Calculate $t^2 * t^3$

Section 7.8: Convolution

Properties of Convolution

Properties of Convolution

Theorem 10. Let $f(t)$, $g(t)$, and $h(t)$ be piecewise continuous on $[0, \infty)$. Then

$$f * g = g * f, \tag{4}$$

$$f * (g + h) = (f * g) + (f * h), \tag{5}$$

$$(f * g) * h = f * (g * h), \tag{6}$$

$$f * 0 = 0. \tag{7}$$

Proof of (5):

Section 7.8: Convolution

Convolution Theorem

Convolution Theorem

Theorem 11. Let $f(t)$ and $g(t)$ be piecewise continuous on $[0, \infty)$ and of exponential order α and set $F(s) = \mathcal{L}\{f\}(s)$ and $G(s) = \mathcal{L}\{g\}(s)$. Then

(8) $$\mathcal{L}\{f * g\}(s) = F(s)G(s),$$

or, equivalently,

(9) $$\mathcal{L}^{-1}\{F(s)G(s)\}(t) = (f * g)(t).$$

Section 7.8: Convolution

Convolution Theorem

$$\mathscr{L}\{f * g\}(s) = F(s)G(s) \tag{8}$$

Proof:

Section 7.8: Convolution

Convolution Theorem

Example 1 Use the convolution theorem to solve the initial value problem

(11) $$y'' - y = g(t)\ ; \qquad y(0) = 1\ , \qquad y'(0) = 1\ ,$$

where $g(t)$ is piecewise continuous on $[0, \infty)$ and of exponential order.

Section 7.8: Convolution

Convolution Theorem

Example 2 Use the convolution theorem to find $\mathscr{L}^{-1}\{1/(s^2+1)^2\}$.

Section 7.8: Convolution

Convolution Theorem

Example 3 Solve the integro-differential equation

$$(12) \qquad y'(t) = 1 - \int_0^t y(t-v)e^{-2v}\,dv\,, \qquad y(0) = 1\,.$$

Section 7.8: Convolution

Goal: Solve the IVP $ay'' + by' + cy = g;\quad y(0) = y_0\,,\quad y'(0) = y_1$ and writing the answer as a convolution

Step 1: Solve the IVP $ay'' + by' + cy = g;\quad y(0) = 0\,,\quad y'(0) = 0$ using the impulse response function

i) The **transfer function** $H(s)$ of a linear system is defined as the ratio of the Laplace transform of the output function $y(t)$ to the Laplace transform of the input function $g(t)$, under the assumption that all initial conditions are zero. That is, $H(s) = Y(s)/G(s)$.

Section 7.8: Convolution

Goal: Solve the IVP $ay'' + by' + cy = g$; $y(0) = y_0$, $y'(0) = y_1$ and writing the answer as a convolution

Step 1: Solve the IVP $ay'' + by' + cy = g$; $y(0) = 0$, $y'(0) = 0$ using the impulse response function

i) For this IVP, the transfer function is $H(s) = \frac{1}{as^2+bs+c}$

ii) The function $h(t) := \mathcal{L}^{-1}\{H\}(t)$ is called the **impulse response function** for the system because, physically speaking, it describes the solution when a mass–spring system is struck by a hammer (see Section 7.9). We can also characterize $h(t)$ as the unique solution to the homogeneous problem

(16) $$ah'' + bh' + ch = 0; \quad h(0) = 0, \quad h'(0) = 1/a.$$

Section 7.8: Convolution

Goal: Solve the IVP $ay'' + by' + cy = g;\quad y(0) = y_0\,,\ \ y'(0) = y_1$ and writing the answer as a convolution

Step 1: Solve the IVP $ay'' + by' + cy = g;\quad y(0) = 0\,,\ \ y'(0) = 0$ using the impulse response function

i) For this IVP, the transfer function is $H(s) = \frac{1}{as^2+bs+c}$

ii) Find the impulse response function $h(t)$ by either calculating $\mathcal{L}^{-1}\{\,H(s)\,\}$ or solving the IVP
$ah'' + bh' + ch = 0;\quad h(0) = 0\,,\quad h''(0) = \frac{1}{a}$

iii) Then the solution to the IVP $ay'' + by' + cy = g;\quad y(0) = 0\,,\ \ y'(0) = 0$ is $(h * g)(t)$

Section 7.8: Convolution

Goal: Solve the IVP $ay'' + by' + cy = g; \quad y(0) = y_0 , \quad y'(0) = y_1$ and writing the answer as a convolution

Step 1: Solve the IVP $ay'' + by' + cy = g; \quad y(0) = 0 , \quad y'(0) = 0$ using the impulse response function

iii) Then the solution to the IVP $ay'' + by' + cy = g; \quad y(0) = 0 , \quad y'(0) = 0$ is $(h * g)(t)$

Step 2: Solve the IVP $ay'' + by' + cy = 0; \quad y(0) = y_0 , \quad y'(0) = y_1$ call the solution y_k
(2^{nd} order homogeneous linear equation section 4.2-4.3 stuff)

Step 3: Then the solution to the IVP $ay'' + by' + cy = g; \quad y(0) = y_0 , \quad y'(0) = y_1$ is

$$\boldsymbol{y(t) = (h * g)(t) + y_k(t)}$$

Section 7.8: Convolution

Solution Using Impulse Response Function

Theorem 12. Let I be an interval containing the origin. The unique solution to the initial value problem

$$ay'' + by' + cy = g\,; \qquad y(0) = y_0\,, \qquad y'(0) = y_1\,,$$

where a, b, and c are constants and g is continuous on I, is given by

$$y(t) = (h * g)(t) + y_k(t) = \int_0^t h(t-v)g(v)\,dv + y_k(t)\,, \tag{20}$$

where h is the impulse response function for the system and y_k is the unique solution to (19).

Section 7.8: Convolution

Example 4 A linear system is governed by the differential equation

$$(21) \qquad y'' + 2y' + 5y = g(t)\,; \qquad y(0) = 2\,, \qquad y'(0) = -2\,.$$

Find the transfer function for the system, the impulse response function, and a formula for the solution.